Measuring the Stokes polarization parameters

Beth Schaefer, Edward Collett, Robert Smyth, Daniel Barrett, and Beth Fraher
Department of Physics, Georgian Court University, 900 Lakewood Avenue, Lakewood, New Jersey 08701-2697

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The Stokes formulation for representing polarized light is discussed along with the classical measurement method for determining the Stokes polarization parameters. The limitations of this method are noted, and we consider the rotating quarter-waveplate method, which avoids the limitations of the classical method and allows a curve fitting algorithm to be used to determine the Stokes parameters. The quarter-waveplate method is attractive because of its accuracy and simplicity and can be implemented in an undergraduate or graduate optics course with a minimum amount of equipment. © *2007 American Association of Physics Teachers.*
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I. INTRODUCTION

In 1852 George Gabriel Stokes showed that the polarization state of light can be characterized in terms of four intensity parameters.[1,2] The Stokes polarization parameters are widely used to describe the polarization behavior of an optical beam[3–5] due primarily to the fact that the polarization ellipse,[6,7] which is an amplitude description of polarized light, is not directly accessible to measurement.[2] The polarization ellipse and its associated orientation and ellipticity angles can be shown to be directly related to the Stokes polarization parameters.[3]

Although several measurement methods can be used to determine the Stokes polarization parameters, our interest is in a method that can be readily implemented in an undergraduate or graduate optics laboratory. To understand this measurement method we first present some background information on the Stokes polarization parameters.

II. THE POLARIZATION ELLIPSE AND THE STOKES POLARIZATION PARAMETERS

The Stokes parameters describe not only completely polarized light (elliptically polarized light), but unpolarized light and partially polarized light as well. Polarization arises when the optical beam consists of two independent orthogonal components, $E_x(z,t)$ and $E_y(z,t)$, that have different amplitudes and phases. The orthogonal components can be taken to be along the x- and y- axes so that the direction of propagation is in the z-direction.[8] The equations that describe this behavior are

$$E_x(z,t) = E_{0x}\cos(\omega t - \kappa z + \delta_x), \tag{1a}$$

$$E_y(z,t) = E_{0y}\cos(\omega t - \kappa z + \delta_y), \tag{1b}$$

where t represents the time. E_{0x} and E_{0y} are the maximum amplitudes of the optical field, $\omega=2\pi\nu$ is the angular frequency, $\kappa=2\pi/\lambda$ is the wave number, and δ_x and δ_y are phase constants. The term $\omega t-\kappa z$ describes the propagation of the wave and is called the propagator. At optical frequencies the duration for a single wave to repeat its oscillation is of the order of 10^{-15} s. This time interval is for all practical purposes immeasurable and so the frequency of oscillation cannot be directly observed or measured.

Because the field components cannot be directly observed, a useful visual representation of the polarization behavior of the optical beam can be obtained by eliminating the propagator in Eq. (1) between the equations for $E_x(z,t)$ and $E_y(z,t)$ yielding

$$\frac{E_x(z,t)^2}{E_{0x}^2} + \frac{E_y(z,t)^2}{E_{0y}^2} - \frac{2E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta, \tag{2}$$

where $\delta=\delta_y-\delta_x$. Equation (2) is called the *polarization ellipse* (see Fig. 1) because it describes the polarization of the optical field.

The ellipse described by Eq. (2) can neither be observed nor measured. To determine the parameters of the polarization ellipse that can be observed, Eq. (2) must be transformed to the intensity (observable) domain.[6] This transformation can be done by taking a time average of Eq. (2). The time average of the quadratic field components, $\langle E_i(z,t)E_j(z,t)\rangle$, is defined by

$$\langle E_i(z,t)E_j(z,t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T E_i(z,t)E_j(z,t)\,dt \quad (i,j=x,y). \tag{3}$$

The time average of Eq. (2) leads to the relation

$$S_0^2 = S_1^2 + S_2^2 + S_3^2, \tag{4}$$

where

$$S_0 = E_{0x}^2 + E_{0y}^2, \tag{5a}$$

$$S_1 = E_{0x}^2 - E_{0y}^2, \tag{5b}$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta, \tag{5c}$$

$$S_3 = 2E_{0x}E_{0y}\sin\delta. \tag{5d}$$

Equation (5) defines the four *Stokes polarization parameters*. They are described in terms of intensities (amplitudes squared) and therefore can be measured. The parameter S_0 describes the total intensity of the optical field, S_1 describes the preponderance of linearly horizontally polarized light (LHP) over linearly vertically polarized light (LVP), S_2 describes the preponderance of linear +45° polarized light (L +45P) over linear −45° polarized light (L−45P), and the fourth parameter S_3 describes the preponderance of right circularly polarized light (RCP) over left circularly polarized light (LCP). The Stokes parameters in Eq. (5) can be arranged as the elements of a 4×1 matrix as

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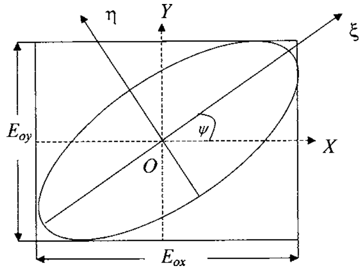

Fig. 1. Plot of the equation for the polarization ellipse, Eq. (2). In general, the ellipse is not in its standard form, where $E_x(z,t)$ and $E_y(z,t)$ are directed along the x- and y-axes, but along an axis rotated through an angle ψ.

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}. \tag{6}$$

Equation (6) is usually called the Stokes vector. In particular, Eq. (6) is the Stokes vector for elliptically (completely) polarized light (ELP). Any state of completely polarized light can be described using Eq. (6).

A set of polarization states that are very useful are the degenerate polarization states,

$$S_{\mathrm{LHP}} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad S_{\mathrm{LVP}} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

$$S_{\mathrm{L+45P}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad S_{\mathrm{L-45P}} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \tag{7}$$

$$S_{\mathrm{RCP}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad S_{\mathrm{LCP}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

The Stokes vectors have been normalized to unit intensity in Eq. (7).

The Stokes parameters describe not only completely polarized light but also unpolarized and partially polarized light as well. To describe these additional polarization states Eq. (4) must be modified as follows:

$$S_0^2 \geqslant S_1^2 + S_2^2 + S_3^2. \tag{8}$$

In Eq. (8) the equality applies to completely polarized light and the inequality applies to both unpolarized and partially polarized light. The Stokes vectors for unpolarized light and partially polarized light are represented by

$$S_{\mathrm{UNP}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad S_{\mathrm{PP}} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, \tag{9}$$

where the subscript UNP and PP refer to unpolarized and partially polarized light, respectively.

Another important measure is the degree of polarization represented by P. For completely polarized light $\mathrm{P}=1$ and for unpolarized light $\mathrm{P}=0$. Partially polarized light is intermediate to these two limits and is represented by $0<\mathrm{P}<1$. The degree of polarization is defined by[3,7]

$$\mathrm{P} = \frac{I_{\mathrm{ELP}}}{I_{\mathrm{TOT}}} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \quad (0 \leqslant \mathrm{P} \leqslant 1). \tag{10}$$

Partially polarized light can be represented as a mixture of unpolarized light (UNP) and completely polarized light. The Stokes vector for partially polarized light can be written as

$$S = \begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix}_{\mathrm{PP}} = (1-\mathrm{P}) \begin{pmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\mathrm{UNP}} + \mathrm{P} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\mathrm{ELP}}. \tag{11}$$

The primes are placed on the Stokes vector S for partially polarized light to emphasize that it is different than that defined for elliptically polarized light. For $\mathrm{P}=0$, Eq. (11) reduces to unpolarized light and for $\mathrm{P}=1$ the equation reduces to elliptically (completely) polarized light.

In many measurements the light is found to be partially polarized and we are interested in determining the completely polarized component. To do so, the four Stokes parameters are measured, and the degree of polarization is found using Eq. (10). If $0<\mathrm{P}<1$, the completely polarized component is found from Eq. (11) to be

$$S_{\mathrm{ELP}} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\mathrm{ELP}} = \frac{1}{\mathrm{P}} \begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix}_{\mathrm{PP}} - \frac{1-\mathrm{P}}{\mathrm{P}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\mathrm{UNP}} \tag{12}$$

$$(0 \leqslant \mathrm{P} \leqslant 1)$$

The polarization ellipse can be described in terms of two angles, the orientation angle ψ and ellipticity angle χ. These angles can be determined by measuring the four Stokes parameters and are given by[3]

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{S_2}{S_1}\right) \quad (0 < \psi \leqslant \pi), \tag{13a}$$

$$\chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3}{S_0}\right) \quad \left(-\frac{\pi}{4} < \chi \leqslant \frac{\pi}{4}\right). \tag{13b}$$

Equation (13) applies to completely polarized light.

III. CLASSICAL MEASUREMENT OF THE STOKES POLARIZATION PARAMETERS

The classical measurement of the Stokes parameters is carried out by allowing the polarized beam to propagate sequentially through two polarizing elements known as a quarter-

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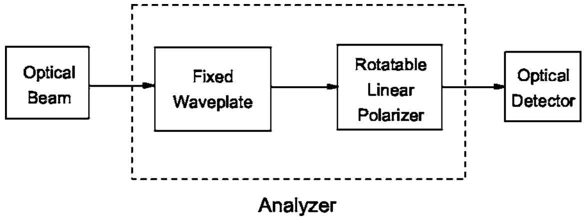

Fig. 2. The classical measurement setup of the Stokes polarization parameters of an optical beam.

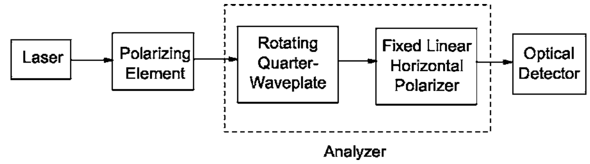

Fig. 3. The rotating quarter-waveplate method for measuring the Stokes polarization parameters. The figure shows the polarizing elements that generate the Stokes parameters to be measured.

waveplate and a linear polarizer.[3,7] A waveplate can be represented by two orthogonal axes known as the fast and slow axes. The waveplate creates a phase shift ϕ between the orthogonal components of a polarized beam. The quarter-waveplate introduces a phase shift of one-quarter of a wave ($\pi/2$ rad) between the orthogonal components of the polarized beam. A linear polarizer is also characterized by a pair of orthogonal transmission axes. Ideally, no transmission occurs along one axis and there is complete transmission along the orthogonal axis. A linear polarizer also has the property that any polarization state of incident light is transformed to linearly polarized light.

In the measurement the waveplate is fixed with its fast axis along the x-axis and the transmission axis of the linear polarizer is rotated through an angle θ. Figure 2 shows the measurement configuration using a polarized laser as an optical source.

The intensity of the optical beam on the detector is given by[3,7]

$$I(\theta,\phi) = \frac{1}{2}(S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta \cos \phi - S_3 \sin 2\theta \sin \phi), \tag{14}$$

where θ refers to the linear polarizer angle and ϕ refers to the phase of the waveplate. The Stokes parameters can be measured by first removing the waveplate and then measuring sequentially the intensity of the optical beam with the linear polarizer set at $\theta=0°$, 45°, and 90°, respectively. The final (fourth) measurement is made by inserting a quarter-waveplate ($\phi=90°$) into the optical train with the linear polarizer set at $\theta=45°$. From Eq. (14) the four Stokes parameters are then found to be[7]

$$S_0 = I(0°,0°) + I(90°,0°), \tag{15a}$$

$$S_1 = I(0°,0°) - I(90°,0°), \tag{15b}$$

$$S_2 = 2I(45°,0°) - S_0, \tag{15c}$$

$$S_3 = S_0 - 2I(45°,90°). \tag{15d}$$

Although this method is satisfactory, it has several drawbacks. It requires that the transmission axis of the linear polarizer be aligned as accurately as possible to the required angles. In the final measurement the quarter waveplate must be inserted and aligned. The introduction of the quarter-waveplate absorbs light, which modifies the equations. Finally, only four data points are measured, which increases the chance of error.

IV. THE ROTATING QUARTER-WAVEPLATE MEASUREMENT

An alternative to the classical measurement of the Stokes parameters is to use a quarter-waveplate that can be rotated through an angle θ; the quarter-waveplate is followed by a fixed linear polarizer whose transmission axis is fixed in the direction of the x-axis. The measurement configuration is shown in Fig. 3 where the optical source is a polarized laser.

An analysis of Fig. 3 shows that the intensity of the optical beam on the detector is given by[7]

$$I(\theta) = \frac{1}{2}(S_0 + S_1 \cos^2 2\theta + S_2 \cos 2\theta \sin 2\theta + S_3 \sin 2\theta). \tag{16}$$

The Stokes parameters in Eq. (16) correspond to the optical beam that emerges from the polarizing element(s). The squared and product terms in Eq. (16) can be rewritten by using the trigonometric half-angle formula to yield[7,9]

$$I(\theta) = \frac{1}{2}(A + B \sin 2\theta + C \cos 4\theta + D \sin 4\theta), \tag{17}$$

where

$$A = S_0 + \frac{S_1}{2}, \quad B = S_3, \quad C = \frac{S_1}{2}, \quad D = \frac{S_2}{2}. \tag{18}$$

Equation (17) is a truncated Fourier series consisting of a dc (constant) term, a second harmonic term, and two fourth harmonic terms. The question arises as to how many data points must be taken to determine A, B, C, and D. The answer is provided by Nyquist's sampling theorem, which states that "a continuous time signal can be reconstructed from its samples if it is sampled at a rate at least twice its highest frequency component."[10] The maximum frequency corresponds to 4θ ($\theta=2\pi ft$) so that to determine A, B, C, and D the minimum number of data points that must be taken is 8.

Because discrete intensities are measured, Eq. (17) is rewritten as

$$I_n = \frac{1}{2}(A + B \sin 2\theta_n + C \cos 4\theta_n + D \sin 4\theta_n) \quad (n = 1,2 \ldots, N) \tag{19}$$

$$(N \geqslant 8),$$

where N is an even number. The coefficients A, B, C, and D are determined using familiar methods from Fourier analysis and are given by

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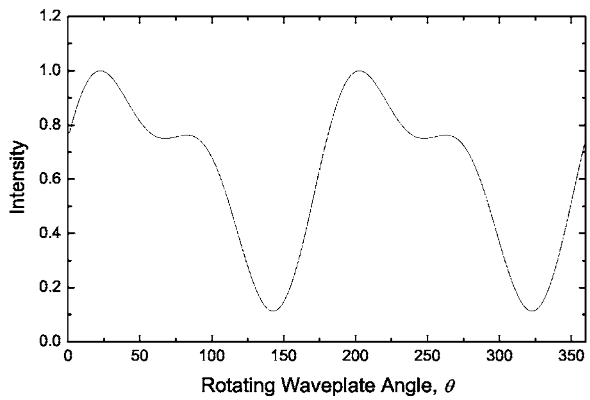

Fig. 4. Intensity plot for Eq. (23) for the continuous rotation of the quarter-waveplate through 360°, showing two identical waveforms; the period of a single waveform is 180° and repeats over 180° intervals.

$$A = \frac{2}{N}\sum_{n=1}^{N} I_n, \quad B = \frac{4}{N}\sum_{n=1}^{N} I_n \sin 2\theta_n, \tag{20a}$$

$$C = \frac{4}{N}\sum_{n=1}^{N} I_n \cos 4\theta_n, \quad D = \frac{4}{N}\sum_{n=1}^{N} I_n \sin 4\theta_n. \tag{20b}$$

In Eq. (20) the intervals are equal and are given by $\theta_{n+1} - \theta_n = 180°/N$. From Eq. (18) the Stokes parameters are found to be

$$S_0 = A - C, \quad S_1 = 2C, \quad S_2 = 2D, \quad S_3 = B. \tag{21}$$

For example, if eight measurements of the normalized intensity at equal intervals are taken and found to be $I_1 = 0.750$ ($\theta_1 = 0.0°$), $I_2 = 1.00$ ($\theta_2 = 22.5°$), $I_3 = 0.854$ ($\theta_3 = 45.0°$), $I_4 = 0.750$ ($\theta_2 = 67.5°$), $I_5 = 0.750$ ($\theta_2 = 90.0°$), $I_6 = 0.500$ ($\theta_2 = 112.5°$), $I_7 = 0.146$ ($\theta_2 = 135.0°$), and $I_8 = 0.250$ ($\theta_2 = 157.5°$), the reader can verify that the intensity equation is

$$I(\theta) = \frac{1}{2}(1.250 + 0.707 \sin 2\theta + 0.250 \cos 4\theta + 0.250 \sin 4\theta). \tag{22}$$

Equation (22) yields the normalized Stokes vector:

$$S = \begin{pmatrix} 1.000 \\ 0.500 \\ 0.500 \\ 0.707 \end{pmatrix}. \tag{23}$$

A plot of the intensity versus the continuous waveplate rotation angle for Eq. (22) is shown in Fig. 4.

V. MEASUREMENTS

We produced different types of polarized light and analyzed it using the rotating quarter-waveplate technique. A rotating half-waveplate will produce various orientations of linearly polarized light. Elliptically polarized light with constant phase can be produced by using a rotating quarter-waveplate (not to be confused with the quarter-waveplate used in the analyzer). Elliptical light of any phase and rotation can be produced by a Babinet-Soleil compensator. The measurement configuration used for these experiments is shown in Fig. 5.

In the measurements the polarizing element was placed in a rotatable mount and then inserted between the light source and the analyzer. A linearly polarized HeNe laser (632.8 nm) was used as the optical source. A calcite polarizer was placed in front of the laser to obtain linearly horizontally polarized light. Only a linearly polarized laser should be used; a randomly polarized laser is not suitable for polarization work.

The intensity is measured by manually rotating the quarter-waveplate through 180° (one-half of a complete rotation) in the analyzer using the PASCO polarization analyzer (OS-8533A) (modified to accommodate the rotating quarter-waveplate) along with the PASCO Rotary Motion Sensor (CI-6538).

The quarter-waveplate only needs to be rotated through 180°, one-half of a complete rotation, to obtain a complete cycle of the intensity. The rotary motion sensor is connected to the rotational mounted quarter-waveplate by means of a tight belt and the resolution was selected to be 1°. This resolution was selected because rotating the quarter-waveplate over a half rotation yields 180 data points, which is a more than sufficient number of data points for analysis.

The output of the motion sensor is a series of digital pulses that indicate the rotation angle of the rotating quarter-waveplate. This angle information is fed to a digital port of the PASCO 750 interface (CI-750) and then to a computer through a SCSI (or USB) port. Simultaneously, the intensity of the beam that is measured by the PASCO optical detector (CI-6504) is also fed to the analog input port of the 750 interface.

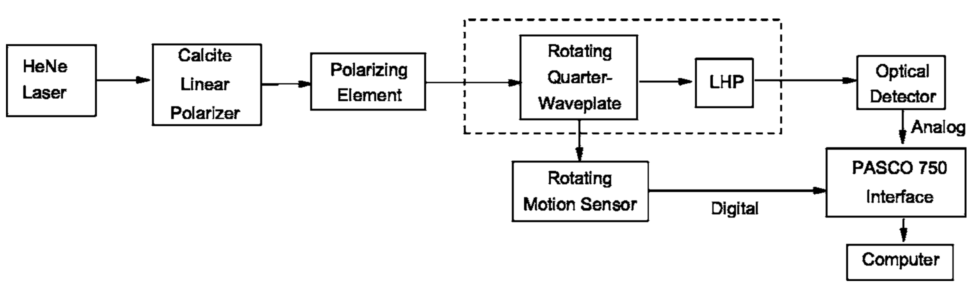

Fig. 5. Setup to measure the Stokes polarization parameters using a rotating quarter-waveplate in the analyzer.

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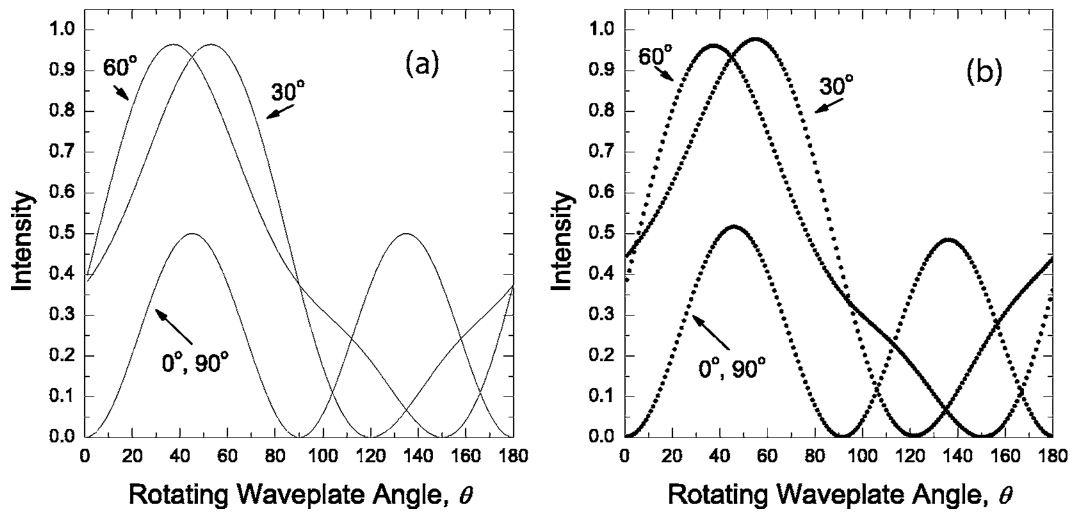

Fig. 6. Intensity plots for a rotated quarter-waveplate. (a) The theoretical curves given by Eq. (34); (b) the measured values.

The data consisted of intensity versus angle scans. A curve fitting routine in Mathematica was used to analyze the data using Eqs. (16), (19), and (20). The routine calculates the four Stokes parameters of the polarized beam from which a continuous curve can then be plotted.

The collected data for both types of measurements were partially polarized. To determine the completely polarized components the measured data were corrected by first finding the degree of polarization, Eq. (10), by using the measured Stokes parameters, and determining the completely polarized component using Eq. (12).

A. Determination of the Stokes parameters for linear polarized light generated by a rotated half-waveplate

In this measurement a half-waveplate was inserted into the optical beam between the laser and analyzer. The half-waveplate must be inserted so that it rotates the light counter-clockwise (looking toward the optical source). For a mechanical rotation of φ with the half-waveplate, the polarization ellipse is rotated by 4φ. For example, if polarized light at a 45° orientation is desired, the half-waveplate must be rotated by 22.5°. A rotator can also be used in place of a half-waveplate, in which case the mechanical rotation angle of α will rotate the polarization ellipse by 2α.

The Stokes vector for the beam that emerges from the rotated half-waveplate is given by

$$S = I_0 \begin{pmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix}, \tag{24}$$

where α is twice the angle φ of rotation of the half-waveplate and I_0 is the intensity of the beam emerging from the laser.

From Eqs. (17) and (24) the intensity of the beam on the optical detector is given by

$$I(\alpha,\theta) = \frac{1}{2} + \frac{1}{2}\cos 2\alpha + \frac{1}{4}\cos(2\alpha - 4\theta), \tag{25}$$

where θ is the angle of rotation of the quarter-waveplate.

The Stokes parameters were calculated from the measured data and agree very well with theory. For example, when the half-waveplate is set to $\alpha=0°$, the Stokes parameters extracted from a representative data scan were (1, 0.999, 0.033, 0.0092), compared with the theoretical value of (1, 1, 0, 0).

B. The Stokes parameters for elliptically polarized light of costant phase generated by a rotated quarter-waveplate

The measurement configuration is identical to that shown in Fig. 5. However, the rotated half-waveplate is now replaced by a rotatable quarter-waveplate (this plate is not to be confused with the quarter-waveplate in the analyzer) with its fast axis along the horizontal x-axis. In addition, the linear polarization of the laser is set to 90°. This configuration will keep the orientation angle of the polarization ellipse positive.

The Stokes vector for the beam that emerges from the rotated quarter-waveplate is given by

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = I_0 \begin{pmatrix} 1 \\ -\cos^2 2\beta \\ -\cos 2\beta \sin 2\beta \\ \sin 2\beta \end{pmatrix}, \tag{26}$$

where β is the angle of rotation of the quarter-waveplate measured counter-clockwise from the horizontal x-axis, and I_0 is the intensity of the beam emerging from the half-waveplate.

From Eq. (17) the intensity of the beam on the optical detector can be shown to be

$$I(\beta,\theta) = \frac{3}{8} - \frac{1}{8}\cos 4\beta + \frac{1}{4}\cos(2\beta - 2\theta) - \frac{1}{4}\cos(2\beta + 2\theta) - \frac{1}{8}\cos(4\beta - 4\theta)\frac{1}{8}\cos 4\theta, \tag{27}$$

where θ is the angle of rotation of the quarter waveplate in the analyzer. Two plots are now made of representative data. Figure 6(a) is the theoretical result given by Eq. (27). Figure 6(b) shows four representative scans of measured values. Both plots include the scans with $\beta=0°,30°,60°,90°$.

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We see that the measured data of the beam emerging from the linear polarizer and the rotated quarter-waveplate agree with the theoretical values. An interesting example occurs when the waveplate is set to $\beta=45°$, which corresponds to right circularly polarized light. The Stokes parameters extracted from a representative data scan were (1, 0.015, 0.0424, 0.999), compared with the theoretical value of (1, 0, 0, 1).

C. The Stokes parameters for elliptically polarized light generated by a Babinet-Soleil compensator

Finally, we point out an interesting behavior of the quarter-waveplate. It is well known that a quarter-waveplate can transform linearly polarized light to circularly polarized light. Less well known is the fact that a rotating quarter-waveplate has the following interesting property. The orientation and ellipticity angles, ψ and χ, are given by [see Eq. (13)]

$$\tan 2\psi = \frac{S_2}{S_1}, \tag{28a}$$

$$\sin 2\chi = \frac{S_3}{S_0}. \tag{28b}$$

From Eqs. (26) and (28) we see that

$$\tan 2\psi = \frac{-\cos 2\beta \sin 2\beta}{-\cos^2 2\beta} = \tan 2\beta, \tag{29a}$$

$$\sin 2\chi = \frac{\sin 2\beta}{1} = \sin 2\beta. \tag{29b}$$

We see from Eq. (29) that β can be used to select either the orientation angle or the ellipticity angle of the polarization ellipse, but not both because the phase of the quarter-waveplate is fixed; there is only one degree of freedom, the rotation. If we wish to change both the orientation angle and the ellipticity angles independently, we must also change the phase. This change can be done by using a Babinet-Soleil compensator, a device that can be used to rotate the axes of the waveplate and also vary the phase ϕ. The Stokes vector for the beam that emerges from the Babinet-Soleil compensator can be shown to be for input vertically polarized light[7]

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = I_0 \begin{pmatrix} 1 \\ -\cos^2 2\beta - \sin^2 2\beta \cos\phi \\ -\cos 2\beta \sin 2\beta(1-\cos\phi) \\ \sin 2\beta \sin\phi \end{pmatrix}. \tag{30}$$

We see that the orientation angle and ellipticity angles are

$$\tan 2\psi = \frac{-\cos 2\beta \sin 2\beta(1-\cos\phi)}{-\cos^2 2\beta + \sin^2 2\beta \cos\phi}, \tag{31a}$$

$$\sin 2\chi = \sin 2\beta \sin\phi = \sin 2\beta. \tag{31b}$$

We substitute $\phi=90°$ (quarter-wave condition) into Eq. (31) and obtain the result for a rotated quarter-waveplate, Eq. (29). Equation (31) shows that we can obtain any value of ψ and χ using the Babinet-Soleil compensator; that is, any state of elliptical polarization can be generated by controlling the rotation and phase of the compensator.

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